Subject: Leaving Certificate Maths

Teacher: Mr Murphy

Lesson 23: Complex Numbers II

23.1 Learning Intentions

After this week's lesson you will be able to;

- Use the Conjugate Roots Theorem to solve am equation with some imaginary roots
- Rewrite a complex number into polar form
- Solve complex equations and expand brackets using De Moivre's Theorem

23.2 Specification

- use the Conjugate Root Theorem to find the roots of polynomials
- work with complex numbers in rectangular and polar form to solve quadratic and other equations including those in the form $z^n = a$, where $n \in \mathbf{Z}$ and $z = r(\text{Cos}\,\theta + i\text{Sin}\,\theta)$
- use De Moivre's Theorem
- prove De Moivre's Theorem by induction for $n \in \mathbf{N}$
- use applications such as n^{th} roots of unity, $n \in \mathbf{N}$, and identities such as $\text{Cos}\,3\theta = 4\,\text{Cos}^3\,\theta - 3\,\text{Cos}\,\theta$

23.3 Chief Examiner's Report

A	4	13·3	40	9	Complex numbers in fraction form

23.4 Equating Complex Numbers

As with algebra, if we have an equals sign, then either side of the equation must be equal. What this means in terms of complex numbers is, the real part on the LHS must equal the real part on the RHS and the same is true for the imaginary part also. For example:

$$a + bi = x + yi$$

In this instance $a = x$ and $b = y$. This can be useful for finding the value of an unknown in an equation involving one or more complex numbers.

Question:

If $z_1 = -3 + 4i$ and $z_2 = 1 + 2i$, given that $z_1 + (a + bi)z_2 = 0$, find the values of a and b given that they are real numbers.

23.5 Conjugate Root Theorem

This theorem states that if z is a root of a quadratic equation then $\bar{z}$ is the other root. Provided the coefficients are real numbers. This can be used with cubic equations that have complex roots $(b^2 - 4ac < 0)$.

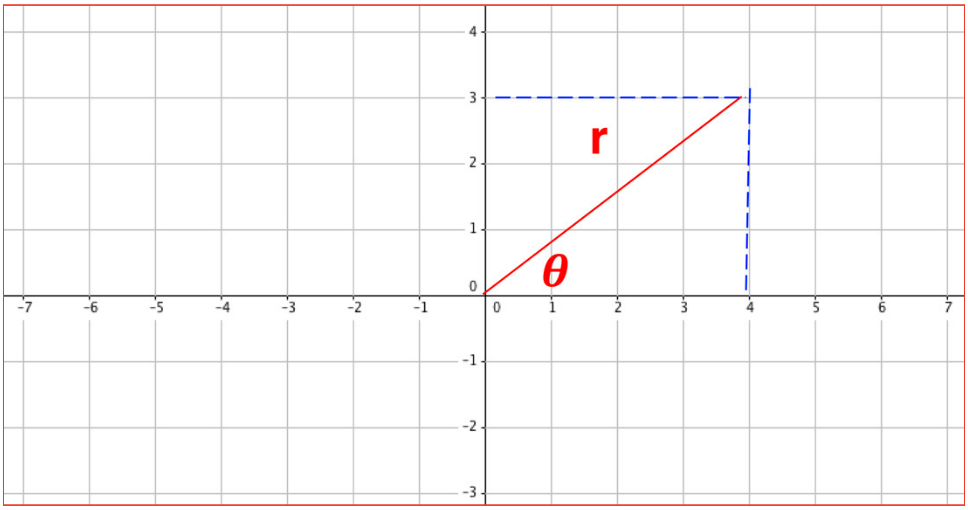
Question:

If $2 + 3i$ is one root of $2z^3 - 9z^2 + 30z - 13 = 0$ find the other two roots

(a) $3 + 2i$ is a root of $z^2 + pz + q = 0$, where $p, q \in \mathbb{R}$, and $i^2 = -1$.
Find the value of p and the value of q.

23.6 Polar Form

Polar form allows us to change our view of the plane our complex number is plotted on. It allows for us to view the plane as a circle. So instead of using horizontal and vertical movements we instead establish the radius of a circle and the angle in which we rotate to find the location of the complex number.



R, we know from lesson 22 is the modulus of the complex number and can be established by:

$$\boldsymbol{r = \sqrt{x^2 + y^2}}$$

θ is referred to as the argument of the number which is established through right angled triangles and the trigonometric ratios.

Let's look at how we might investigate how to calculate this angle:

23.7 De Moivre's Theorem

This can be useful in a number of areas. Where you will come across it most is in expanding brackets or solving equations involving complex numbers. Let's look at expanding brackets first.

Using your knowledge of expanding brackets from algebra, expand $(3 - 4i)^3$.

What if this was $(3 - 4i)^6$?

This would be very time consuming. If we convert our complex number into polar form, we have a much easier way or dealing with this. Based on our laws in algebra we can do the following:

$(3 - 4i)^6$

That was expanding brackets, now we will look at solving equations using complex numbers raised to a power, normally a high enough power.

Let us try to solve the following equation using our newfound knowledge:

Solve: $z^3 = 8i$

(b) **(i)** $v = 2 - 2\sqrt{3}i$. Write v in the form $r(\cos\theta + i\sin\theta)$, where $r \in \mathbb{R}$ and $0 \le \theta \le 2\pi$.

(ii) Use your answer to **part (b)(i)** to find the **two** possible values of w, where $w^2 = v$. Give your answers in the form $a + ib$, where $a, b \in \mathbb{R}$.

23.8 Recap of the Learning Intentions

After this week's lesson you will be able to;

- Use the Conjugate Roots Theorem to solve am equation with some imaginary roots
- Rewrite a complex number into polar form
- Solve complex equations and expand brackets using De Moivre's Theorem

23.9 Homework Task

$z = -\sqrt{3} + i$, where $i^2 = -1$.

(a) Use De Moivre's Theorem to write z^4 in the form $a + b\sqrt{c}\,i$, where a, b, and $c \in \mathbb{Z}$.

(b) The complex number w is such that $|w| = 3$ and w makes an angle of $30°$ with the positive sense of the real axis. If $t = zw$, write t in its simplest form.

23.10 Solutions to 22.11

(a) The complex numbers z_1, z_2 and z_3 are such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$, $z_2 = 2 + 3i$ and $z_3 = 3 - 2i$, where $i^2 = -1$. Write z_1 in the form $a + bi$, where $a, b \in \mathbb{Z}$.

$$\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$$

$$\frac{2}{z_1} = \frac{1}{2+3i} + \frac{1}{3-2i}$$

$$\frac{2}{z_1} = \frac{3-2i+2+3i}{(2+3i)(3-2i)}$$

$$\frac{2}{z_1} = \frac{5+i}{12+5i}$$

$$\frac{z_1}{2} = \frac{12+5i}{5+i}$$

$$\frac{z_1}{2} = \frac{12+5i}{5+i} \times \frac{5-i}{5-i}$$

$$\frac{z_1}{2} = \frac{65+13i}{26}$$

$$z_1 = \frac{130+26i}{26}$$

$$z_1 = 5 + i$$